Optimal and Non-optimal Rates of Approximation for Integrated Semigroups and Cosine Functions

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For a general approximation process we formulate theorems concerning rates of convergence, including theorems about saturation class, non-optimal rates, and sharpness of non-optimal convergence. The general results are then applied to *n*-times integrated semigroups and cosine functions, yielding some new results about their approximation, as well as the convergence of their Cesáro and Abel means to the identity.   © 1997 Academic Press

1. INTRODUCTION

A family $\{S_\rho\}$, $\rho \in (0, \infty)$, of bounded linear operators S_ρ on a Banach space X is called a (uniformly bounded) *strong approximation process* on X if there is a constant M such that $\|S_\rho\| \leqslant M$ for all $\rho > 0$ and

$$(\text{A}_1) \qquad \lim_{\rho \to \infty} \|S_\rho x - x\| = 0 \qquad \text{for all} \quad x \in X.$$

Saturation is an interesting phenomenon in the approximation theory. This concept was introduced by Favard in 1947. The process $\{S_\rho\}$ is said to possess the *saturation property* if there exists a positive function $\phi(\rho)$ tending monotonically to zero as $\rho \to \infty$ such that every $x \in X$ for which

$$\|S_\rho x - x\| = \circ(\phi(\rho)) \qquad (\rho \to \infty)$$

is an invariant element of $\{S_\rho\}$, i.e., $S_\rho x = x$ for all $\rho \in (0, \infty)$, and if the set

$$F[X; S_\rho] = \{x \in X; \|S_\rho x - x\| = O(\phi(\rho))(\rho \to \infty)\}$$

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contains at least one noninvariant element. In this event, the approximation process $\{S_\rho\}$ is said to have *optimal approximation order* $O(\phi(\rho))$ or to be *saturated with order* $O(\phi(\rho))$, and $F[X; S_\rho]$ is called its *Favard class or saturation class.* See e.g. [6, p. 434] for the above definitions.

Thus the saturation concept consists of determination of the optimal order $O(\phi(\rho))$ of approximation and the class of elements which can be approximated with this optimal order. This problem has been investigated by many authors. In particular, we mention the following theorem of Butzer and Nessel [6, Theorem 13.4.1, p. 502], which discusses saturation under two further assumptions:

(A_2)   there are a densely defined closed operator B and a positive number α such that

$$\lim_{\rho \to \infty} \| \rho^\alpha [S_\rho x - x] - Bx \| = 0$$

for every $x \in D(B)$;

(A_3)   there is a regularization process, i.e., a family of bounded operators $\{J_n\}$, $n \in N$, from X into X such that the range of J_n is contained in $D(B)$ for each $n \in N$, $\lim_{n \to \infty} \| J_n x - x \| = 0$ for each $x \in X$, and operators J_n and S_ρ are commutative for all $n \in N$ and $\rho > 0$.

THEOREM A.   *Let X be a Banach space, $\{S_\rho\}$ be a strong approximation process on X which satisfies conditions* (A_1), (A_2), *and* (A_3). *Then we have*:

(i)   *If $x \in X$ is such that $\| S_\rho x - x \| = \circ(\rho^{-\alpha})$, then $x \in D(B)$ and $Bx = 0$.*

(ii)   *The following conditions are equivalent*:

(a) $\| S_\rho x - x \| = O(\rho^{-\alpha})(\rho \to \infty)$;

(b) $x \in \widetilde{D(B)}^X$;

(c) $x \in D(B)$, *provided X is reflexive.*

Here $D(B)$ is the Banach space with the graph norm $\| \cdot \|_{D(B)}$ of B and $\widetilde{D(B)}^X :=$ $\{x \in X; \exists\{x_n\} \subset D(B)$ such that $\lim_{n \to \infty} \| x_n - x \| = 0$ and $\sup \| x_n \|_{D(B)} < \infty \}$.

Thus the Favard class is characterized with the completion of $D(B)$ relative to X. Application of Theorem A to a C_0-semigroup $T(\cdot)$ with generator A yields the saturation theorem (see [6, Theorem 13.4.4, p. 505] and [3, Theorem 2.1.2, p. 88, and Prop. 2.3.1, p. 111]) that, as $t \to 0$, $\| T(t) x - x \| = O(t)$ (resp., $o(t)$) if and only if $x \in \widetilde{D(A)}^X$ (resp. $x \in N(A)$). Butzer and Dickmeis [4] proved that if A possesses a sequence $\{\lambda_n\}$ of

eigenvalues with $|\lambda_n| \to \infty$, then for each $0 < \beta < 1$ there exists an $x^* \in X$ such that

$$\|T(t) x^* - x^*\| \begin{cases} = O(t^\beta) \\ \neq \circ(t^\beta) \end{cases} \qquad (t \to 0^+).$$

Davydov [9] further proved that the sharpness of non-optimal approximation holds true for any semigroup having an unbounded generator.

The aim of this paper is to consider the "non-optimal" approximation of $\{S_\rho\}$. Using a K-functional we give necessary and sufficient conditions upon an element $x \in X$ such that

$$\|S_\rho x - x\| = O(\rho^{-\beta}) \qquad (0 < \beta < \alpha, \rho \to \infty).$$

Moreover, it is proved that if B is unbounded, then there exists an $x^* \in X$ such that

$$\|S_\rho x^* - x^*\| \begin{cases} = O(\rho^{-\beta}) \\ \neq \circ(\rho^{-\beta}) \end{cases} \qquad (\rho \to \infty).$$

The proof is based on a deep fundamental result of Davydov [9].

In Section 2, we prove general results on rates and sharpness of non-optimal convergence for a general approximation process, and then, in Sections 3 and 4, we apply them to n-times integrated semigroups and n-times integrated cosine functions. Note that from the general results in Section 2 one can also deduce our recent results [8, Theorems 2.3 and 2.4] on optimal and on-optimal rates of approximation for resolvent families. The particular results for resolvent families and for n-times integrated semigroups and cosine functions all generalize the corresponding results (see [2–7, 9]) for C_0-semigroups and cosine operator functions.

2. STRONG APPROXIMATION WITH RATES

For convenience, we first observe some properties of $\{S_\rho\}$, $\{J_n\}$, and B in Theorem A. Since B is a closed operator, $D(B)$ becomes a Banach space with the graph norm $\|x\|_{D(B)} := \|x\| + \|Bx\|$ for $x \in D(B)$. $\{\rho^\alpha(S_\rho - I)|_{D(B)}\}$ is a family of bounded linear operators from $D(B)$ into X. Using (A_2), for each $x \in D(B)$ there exists a constant $M_x > 0$ such that $\rho^\alpha \|S_\rho x - x\| \leq M_x$ for all $\rho \geq 0$. Hence, by the principle of uniform boundedness there exists M_1 such that

$$\rho^\alpha \|S_\rho x - x\| \leq M_1 \|x\|_{D(B)} \qquad (x \in D(B), \rho > 0). \tag{1}$$

Similarly, by (A_3) and the principle of uniform boundedness there also exists $M_2 > 0$, independent of x and n, such that

$$\|J_n x\| \leqslant M_2 \|x\| \qquad \text{for all} \quad x \in X. \tag{2}$$

Now, we consider the non-optimal approximation of $\{S_\rho\}$. Under some additional assumptions we find necessary and sufficient conditions upon an element $x \in X$ such that

$$\|S_\rho x - x\| = O(\rho^{-\beta}) \qquad (\rho \to \infty),$$

where $0 < \beta < \alpha$. Before doing this, we recall the definition of a K-functional.

DEFINITION 2.1.   Let X be a Banach space with norm $\|\cdot\|_X$ and Y be a submanifold with seminorm $\|\cdot\|_Y$. The K-functional is defined by

$$K(t, x) := K(t, x, X, Y, \|\cdot\|_Y) = \inf_{y \in Y} \{\|x - y\|_X + t \|y\|_Y\}.$$

If Y is also a Banach space with $\|\cdot\|_Y$, then the completion of Y relative to X is defined as

$$\tilde{Y}^X := \{x \in X; \exists \{x_n\} \subset Y \text{ such that } \lim_{n \to \infty} \|x_n - x\|_X = 0$$

$$\text{and } \sup \|x_n\|_Y < \infty\}.$$

It is well known that $K(t, x)$ is a bounded, continuous, monotone increasing, and subadditive function of t for each $x \in X$ (cf. [2] and [3]). By using a K-functional we can give the following characterization of non-optimal convergence of $\{S_\rho x\}$.

THEOREM 2.2.   Suppose $\{S_\rho\}$, $\{J_n\}$, and B, as defined in Theorem A, satisfy conditions (A_1)–(A_3), and also satisfy the following condition for $x \in X$ and $0 < \beta \leqslant \alpha$:

(A_4) If $\|S_\rho x - x\| = O(\rho^{-\beta})(\rho \to \infty)$, then there are $\rho_0 > 0$, C_1 and C_2, depending on x and β, such that for each $\rho \geqslant \rho_0$ there exists an n_ρ for which $\|B J_{n_\rho} x\| \leqslant C_1 \rho^{\alpha - \beta}$ and $\|J_{n_\rho} x - x\| \leqslant C_2 \rho^{-\beta}$.

Then

$$\|S_\rho x - x\| = O(\rho^{-\beta}) \qquad \text{if and only if}$$

$$K(\rho^{-\alpha}, x, X, D(B), \|\cdot\|_{D(B)}) = O(\rho^{-\beta}).$$

Proof. (Sufficiency) Using (1) we have for any $y \in D(B)$ and $\rho > 0$

$$\|S_\rho x - x\| \leqslant \|(S_\rho - I)(x - y)\| + \|S_\rho y - y\|$$
$$\leqslant (M + 1) \|x - y\| + \rho^{-\alpha} M_1 \|y\|_{D(B)}$$
$$\leqslant \max(M + 1, M_1)[\|x - y\| + \rho^{-\alpha} \|y\|_{D(B)}].$$

Hence $\|S_\rho x - x\| \leqslant \max(M + 1, M_1) K(\rho^{-\alpha}, x, X, D(B), \|\cdot\|_{D(B)}) = O(\rho^{-\beta})$.

(Necessity) If $\|S_\rho x - x\| = O(\rho^{-\beta})$, then, by (A_4) and (2), we have for any $\rho \geqslant \rho_0$

$$K(\rho^{-\alpha}, x, X, D(B), \|\cdot\|_{D(B)}) \leqslant \|x - J_{n_p} x\| + \rho^{-\alpha} \|J_{n_p} x\|_{D(B)}$$
$$\leqslant C_2 \rho^{-\beta} + \rho^{-\alpha} M_2 \|x\| + C_1 \rho^{-\alpha} \cdot \rho^{\alpha - \beta}$$
$$\leqslant (C_2 + M_2 \|x\| + C_1) \rho^{-\beta}.$$

Hence $K(\rho^{-\alpha}, x, X, D(B), \|\cdot\|_{D(B)}) = O(\rho^{-\beta})$ $(\rho \to \infty)$.

Remarks. (i) Under assumptions (A_2) and (A_3) we have

$$\|BJ_n x\| = \lim_{\rho \to \infty} \rho^\alpha \|S_\rho J_n x - J_n x\| = \lim_{\rho \to \infty} \rho^\alpha \|J_n[S_\rho x - x]\|$$
$$\leqslant M_2 \varlimsup_{\rho \to \infty} \rho^\alpha \|S_\rho x - x\|.$$

Thus, if $\alpha = \beta$, then (A_4) automatically holds. Hence, Theorem A is also a consequence of Theorem 2.2 and a property of the K-functional (see [2]).

(ii) Since $\|x - y\| + \rho^{-\alpha} \|y\|_{D(B)} \leqslant (1 + \rho^{-\alpha})[\|x - y\| + \rho^{-\alpha}\|Bx\|] + \rho^{-\alpha}\|x\|$ for $x \in X$, $y \in D(B)$, Theorem 2.2 still holds if in the definition of the K-functional the graph norm of $D(B)$ is replaced by the seminorm $\|Bx\|$ for $x \in D(B)$.

Moreover, we also can give the sharpness of non-optimal approximation for $\{S_\rho\}$. To do this, we need the following lemma.

LEMMA 2.3. *Suppose* $\{S_\rho\}$, $\{J_n\}$, *and* B *satisfy* (A_1)–(A_3), *and also satisfy the following assumption*:

(A_5) *If* $\lim_{\rho \to \infty} \|S_\rho - I\| = 0$, *then* $\lim_{n \to \infty} \|J_n - I\| = 0$.

Then B *is bounded if and only if* $\|S_\rho - I\| \to 0$. *In this case,* $\|S_\rho - I\| = O(\rho^{-\alpha})$.

Proof. The necessity follows from (A_2) and the uniform boundedness principle. Suppose $\|S_\rho - I\| \to 0$ as $\rho \to \infty$. Define $A_{n,\rho}: X \to X$ by $A_{n,\rho} x := J_n S_\rho x$. $A_{n,\rho}$ is a bounded linear operator with range contained in $D(B)$. By (2) we have

$$\|A_{n,\rho}x - x\| = \|J_n S_\rho x - x\|$$
$$\leqslant \|J_n S_\rho x - J_n x\| + \|J_n x - x\|$$
$$\leqslant M_2 \|S_\rho x - x\| + \|J_n - I\| \|x\|$$
$$\leqslant M_2 \|S_\rho - I\| \|x\| + \|J_n - I\| \|x\|.$$

Hence, by (A_5) there exist n_0 and ρ_0 such that $\|A_{n_0,\rho_0} - I\| \leqslant \frac{1}{2}$. This implies that A_{n_0,ρ_0} is invertible, so that its range is the whole space X. Hence $D(B) = X$, and so B must be bounded.

Note that (A_4) and (A_5) are independent.

THEOREM 2.4. *Suppose* $\{S_\rho\}$, $\{J_n\}$, *and* B *saisfy* (A_2), (A_3), *and* (A_5). *Then* B *is unbounded if and only if for each* $0 < \beta < \alpha$ *there exists* x_β^* *such that*

$$\|S_\rho x_\beta^* - x_\beta^*\| \begin{cases} = O(\rho^{-\beta}) \\ \neq \circ(\rho^{-\beta}) \end{cases} \qquad (\rho \to \infty).$$

If $\beta = \alpha$ and B is not equal to the null operator, then, by Theorem A, there always exist such x_β^*.

To prove this theorem, we need the following theorem which is shown by Davydov [9].

THEOREM 2.5. *Let* X *be a Banach space and* X^+ *be the set of all non-negative, sublinear, real-valued functions* S *on* X *for which the norm* $\|S\|_{X^+} := \sup\{S(x); x \in X, \|x\| \leqslant 1\}$ *is bounded. Further, let* H *be an unbounded set of continuous seminorms and let* $\{x \in X; \lim_{\|h\|_{X^+} \to \infty} h(x) = 0, h \in H\}$ *be dense in* X. *Then there exists an element* $x^* \in X$ *such that* $\sup_{h \in H} h(x^*) \leqslant 1$ *and* $\overline{\lim}_{\|h\|_{X^+} \to \infty} h(x^*) = 1$.

Proof of Theorem 2.4. If B is bounded, by (A_2) and the uniform boundedness principle we have $\|S_\rho - I\| = O(\rho^{-\alpha})$ so that $\|S_\rho - I\| = \circ(\rho^{-\beta})$ $(\rho \to \infty)$ for all $0 < \beta < \alpha$. This shows the sufficiency.

To show the necessity, define $H := \{h_\rho; h_\rho(x) := \rho^\beta \|S_\rho x - x\|, x \in X$ for $\rho > 0\}$. Let us check that H satisfies the hypothesis of Theorem 2.5. By the uniform boundedness of $\{S_\rho\}$, $\|h_\rho\|_{X^+} \leqslant (M+1)\rho^\beta$ for some $M > 0$ and this implies that if $\|h_\rho\|_{X^+} \to \infty$, then $\rho \to \infty$. By (A_2), $D(B) \subset \{x \in X; \lim_{\|h\|_{X^+} \to \infty} h(x) = 0, h \in H\}$ is dense in X. If B is unbounded, by Lemma 2.3 S_ρ is not convergent in operator norm. Hence $\overline{\lim}_{\rho \to \infty} \|S_\rho - I\| > 0$, so that $\lim_{\rho \to \infty} \|h_\rho\|_{X^+} = \lim_{\rho \to \infty} \rho^\beta \|S_\rho - I\| = \infty$. Thus H is an unbounded set which satisfies the hypothesis of Theorem 2.5. Hence there exists $x^* \in X$ such that $\sup_{\rho > 0} h_\rho(x^*) \leqslant 1$ and $\overline{\lim}_{\rho \to \infty} h_\rho(x^*) = 1$, i.e., x^* satisfies $\|S_\rho x^* - x^*\| = O(\rho^{-\beta})$ $(\rho \to \infty)$ but $\|S_\rho x^* - x^*\| \neq \circ(\rho^{-\beta})$ $(\rho \to \infty)$.

3. APPLICATIONS TO *N*-TIMES INTEGRATED SEMIGROUPS

Let X be a Banach space. We denote by $B(X)$ the set of all bounded linear operators. Let n be a natural number. A strongly continuous family $\{T(t);\ t \geqslant 0\}$ in $B(X)$ is called a *n-times integrated semigroup* on X, if $T(0) = 0$ and

$$T(t)\,T(s)\,x = \frac{1}{(n-1)!} \left(\int_t^{t+s} (s+t-r)^{n-1}\,T(r)x\,dr \right.$$
$$\left. - \int_0^s (t+s-r)^{n-1}\,T(r)\,x\,dr \right)$$

for $x \in X$ and $t, s \geqslant 0$. $T(\cdot)$ is *nondegenerate* if $T(t)x = 0$ for all $t > 0$ implies $x = 0$.

For convenience we call a semigroup of class C_0 on X also a 0-time integrated semigroup on X.

The *generator* A of a nondegenerate n-times integrated semigroup $T(\cdot)$ is defined as follows:

$$x \in D(A) \qquad \text{and} \qquad Ax = y \qquad \text{if and only if}$$
$$T(t)x = \int_0^t T(r)\,y\,dr + \frac{t^n}{n!}\,x \qquad \text{for} \quad t \geqslant 0.$$

If there are M and ω such that $\|T(t)\| \leqslant Me^{\omega t}$, then we say that $T(\cdot)$ is *exponentially bounded.* In this case, one has

$$(\omega, \infty) \subset \rho(A) \qquad \text{and} \qquad (\lambda - A)^{-1} x = \int_0^\infty \lambda^n e^{-\lambda t} T(t)x\,dt$$

for $x \in X$ and $\lambda > \omega$. We say $A \in I_n$ if A generates a n-times integrated semigroup $T(\cdot)$ satisfying $\|T(t)\| = O(t^n)$ $(t \to 0^+)$. Note that A may not be densely defined. For basic properties of n-times integrated semigroups see [1, 10, 11, 14, 16, 17, and 19].

DEFINITION 3.1. We call a nondegenerate one-time integrated semigroup $T(\cdot)$ an F_0-semigroup with generator A and write $A \in F_0$ if $T(\cdot)$ satisfies the local Lipschitz continuity:

$$\|T(t) - T(r)\| \leqslant \Lambda_\tau\,|t - r| \qquad \text{for} \quad 0 \leqslant t, r \leqslant \tau < \infty,$$

where Λ_τ is a finite number for each $\tau < \infty$. Clearly, $F_0 \subset I_1$.

DEFINITION 3.2.   Let A be a linear operator defined on X with domain $D(A)$. We say $A \in G(M, \omega)$, if $\lambda I - A$ is invertible for $\lambda > \omega$ and $R(\lambda; A) = (\lambda I - A)^{-1}$ is a bounded linear operator satisfying the Hille–Yosida condition:

$$\|R(\lambda; A)^n\| \leqslant \frac{M}{(\lambda - \omega)^n} \qquad \text{for} \quad \lambda > \omega, \quad n = 1, 2, 3, \dots .$$

We start with a result which determines what elements of X can be approximated by $T(t)$ $(t \to 0^+)$.

THEOREM 3.3.   Let $A \in I_n$ generate a n-times integrated semigroup $T(\cdot)$. Then $(n!/t^n) T(t) x$ converges to x as $t \to 0^+$ if and only if $x \in \overline{D(A)}$.

Proof.   Since $T(t)x \in \overline{D(A)}$ for each $x \in X$ and $t \geqslant 0$, the convergence of $(n!/t^n) T(t) x$ to x implies $x \in \overline{D(A)}$. Conversely, since $\|T(t)\| \leqslant Mt^n$ as $t \to 0^+$ for some $M > 0$, it is sufficient to show that $(n!/t^n) T(t) x$ converges to x as $t \to 0^+$ for $x \in D(A)$. Let $x \in D(A)$. Then

$$\left\| \frac{n!}{t^n} T(t)x - x \right\| = \left\| \frac{n!}{t^n} \int_0^t T(u) Ax \, du \right\| \leqslant \frac{n!}{t^n} \int_0^t \|T(u) Ax\| \, du$$

$$\leqslant \frac{Mn!}{n+1} t \, \|Ax\|$$

for sufficiently small t. Hence, we have shown that $(n!/t^n) T(t) x$ converges to x as $t \to 0^+$ for $x \in D(A)$.

From Theorem 3.3 we can easily show the following corollary.

COROLLARY 3.4.   If $A \in I_n$, then for $x \in \overline{D(A)}$

(a) $\dfrac{1}{t} \displaystyle\int_0^t \frac{n! \, T(u)}{u^n} x \, du \to x$ \qquad as \quad $t \to 0^+$;

(b) $\dfrac{(n+1)!}{t^{n+1}} \displaystyle\int_0^t T(u)x \, du \to x$ \qquad as \quad $t \to 0^+$.

If A is the generator of a n-times integrated semigroup, let B be the part of A in $\overline{D(A)}$, i.e., $Bx := Ax$ for $x \in D(B) = \{x \in D(A); \ Ax \in \overline{D(A)}\}$. We will use this operator to characterize the rates.

LEMMA 3.5. *If $A \in I_n$, then B is densely defined from $\overline{D(A)}$ into $\overline{D(A)}$. Moreover, for $x \in D(B)$ we have*

(a) $\displaystyle \lim_{t \to 0^+} \frac{1}{t} \left(\frac{n!}{t^n} T(t) x - x \right) = \frac{1}{n+1} Bx;$

(b) $\displaystyle \lim_{t \to 0^+} \frac{1}{t} \left(\frac{(n+1)!}{t^{n+1}} \int_0^t T(u)x \, du - x \right) = \frac{1}{n+2} Bx;$

(c) $\displaystyle \lim_{t \to 0^+} \frac{1}{t} \left(\frac{1}{t} \int_0^t \frac{n! \, T(u)x}{u^n} \, du - x \right) = \frac{1}{2(n+1)} Bx.$

Proof. First to show that B is densely defined on $\overline{D(A)}$, let $x \in D(A)$. Then $(n!/t^n) T(t) x \in D(A)$ and $A(n!/t^n) T(t) x = (n!/t^n) T(t) Ax \in \overline{D(A)}$. Hence $n! \, T(t)x/t^n \in D(B)$. Letting $t \to 0^+$, by Theorem 3.3 we obtain $(n!/t^n) T(t)x \to x$ as $t \to 0^+$. It follows that $D(A) \subset \overline{D(B)} \subset \overline{D(A)}$. Then B is densely defined on $\overline{D(A)}$.

To show (a), let $x \in D(B)$. Then $x \in D(A)$ and $Ax \in \overline{(DA)}$ so that

$$\left\| \frac{1}{t} \left(\frac{n! \, T(t)x}{t^n} - x \right) - \frac{1}{n+1} Bx \right\| = \left\| \frac{n!}{t^{n+1}} \int_0^t T(u) \, Ax \, du - \frac{1}{n+1} Ax \right\|$$

$$\leqslant \frac{1}{t^{n+1}} \int_0^t u^n \left\| \frac{n! \, T(u)}{u^n} Ax - Ax \right\| du.$$

From Theorem 3.3, it follows that $\lim_{t \to 0^+} (1/t)((n!/t^n) T(n)x - x) = (1/(n+1)) Bx$.

To show (b), for $x \in D(B)$ we have

$$\left\| \frac{1}{t} \left\{ \frac{(n+1)!}{t^{n+1}} \int_0^t T(u)x \, du - x \right\} - \frac{1}{n+2} Bx \right\|$$

$$= \frac{(n+1)}{t^{n+2}} \left\| \int_0^t \left[n! \, T(u)x - u^n x - \frac{u^{n+1}}{n+1} Bx \right] du \right\|$$

$$\leqslant \frac{(n+1)}{t^{n+2}} \int_0^t \left\| u^{n+1} \left\{ \frac{n! \, T(u)x}{u^{n+1}} - \frac{x}{u} - \frac{1}{n+1} Bx \right\} \right\| du.$$

Hence (b) follows from (a).

To show (c), for $x \in D(B)$ we have

$$\left\| \frac{1}{t} \left\{ \frac{1}{t} \int_0^t \frac{n!\, T(u)x}{u^n}\, du - x \right\} - \frac{Bx}{2(n+1)} \right\|$$

$$\leqslant \frac{1}{t^2} \int_0^t u \left\| \left\{ \frac{n!\, T(u)x}{u^{n+1}} - \frac{x}{u} - \frac{1}{n+1}\, Bx \right\} \right\|\, du.$$

Hence (c) also follows from (a).

LEMMA 3.6.   *Let* $A \in I_n$. *We define the linear operators* $J_{1,m}$ *and* $J_{2,m}$ *on* X *as follows:* $J_{1,m}x := (n+1)!\, m^{n+1} \int_0^{1/m} T(u)x\, du$, $J_{2,m}x := (n+2)!\, m^{n+2} \int_0^{1/m} \int_0^u T(s)x\, ds\, du$. *Then for* $i = 1, 2$,

(a) $J_{i,m}$ *is uniformly bounded;*

(b) $J_{i,m}x \in D(B)$ *for every* $x \in \overline{D(A)}$;

(c) $\lim_{m \to \infty} J_{i,m}x = x$ *for* $x \in \overline{D(A)}$.

Proof. (a) and (b) are obvious. It remains to show (c). For $i = 1$ the proof follows from Corollary 3.4(b). For $i = 2$ and $x \in \overline{D(A)}$ we have

$$\|J_{2,m}x - x\| \leqslant (n+2)\, m^{n+2} \int_0^{1/m} \left\| u^{n+1} \left\{ \frac{(n+1)!}{u^{n+1}} \int_0^u T(s)x\, ds - x \right\} \right\|\, du.$$

Then from Corollary 3.4(b) we derive the result.

If A generates an n-times integrated semigroup $\{T(t);\ t > 0\}$, we know that A may not be densely defined, but $T(t)x$ still belongs to $\overline{D(A)}$ for each $x \in X$. Hence $((n+1)!/t^{n+1}) \int_0^t T(u)x\, du$ and $(n!/t) \int_0^t (1/u^n) T(u)x\, du$ still belong to $\overline{D(A)}$ for each $x \in X$. If we write $S^1 = \{(n!/t^n) T(t),\ J_{1,m},\ 1/(n+1)B\}$, $S^2 = \{(n+1)!/t^{n+1}) \int_0^t T(u)\, du,\ J_{2,m},\ 1/(n+2)B\}$ and $S^3 = \{(n!/t) \int_0^t (1/u^n) T(u)\, du,\ J_{2,m},\ 1/2(n+1)B\}$, then we can summarize Theorem 3.3, Corollary 3.4, and Lemmas 3.5 and 3.6 with the following lemma.

LEMMA 3.7.   *If* $A \in I_n$, *then* S^1, S^2, *and* S^3 *satisfy the hypotheses of Theorem A with parameter* $\rho = 1/t$ *for* $0 < t \leqslant t_0$ *and* $\alpha = 1$ *on the Banach space* $\overline{D(A)}$.

Therefore the next theorem follows immediately from (i) of Theorem A.

THEOREM 3.8. *If* $A \in I_n$, *then the following are equivalent:*

(a) $\left\| \dfrac{n!}{t^n} T(t)x - x \right\| = \circ(t)$ $(t \to 0^+)$;

(b) $\left\| \dfrac{(n+1)!}{t^{n+1}} \int_0^t T(u)x \, du - x \right\| = \circ(t)$ $(t \to 0^+)$;

(c) $\left\| \dfrac{n!}{t} \int_0^t \dfrac{T(u)x}{u^n} \, du - x \right\| = \circ(t)$ $(t \to 0^+)$;

(d) $x \in N(B) = N(A)$, *the null space of* B *and* A.

In the proof of Theorem 3.10 we shall need the following lemma, which generalizes a lemma of van Neerven [13, Lemma 3.3.2] from 0 to general n.

LEMMA 3.9. *Suppose that* A *generates an* n-*times integrated semigroup* $T(\cdot)$ *such that* $\|T(t)\| \leqslant Mt^n$ *for* $t \in (0, t_0)$ *and* $M \geqslant 1$. *Then for* $x \in \overline{D(A)}$ *we have*

$$K(t, x, X, D(A), \|\cdot\|_{D(A)}) \leqslant K(t, x, \overline{D(A)}, D(B), \|\cdot\|_{D(B)})$$
$$\leqslant Mn! \, K(t, x, X, D(A), \|\cdot\|_{D(A)}).$$

Proof. The first inequality is obvious from the definition of the K-functional.

To show the second inequality, fix $\varepsilon > 0$ and $x \in \overline{D(A)}$ arbitrarily. By Corollary 3.4 there exists a sufficiently large m such that $\|x - J_{i,m}x\| \leqslant \varepsilon$.

For this m we consider the map $L_m^1 = J_{i,m}: X \to X$, and the map $L_m^2: (D(A), \|\cdot\|_{D(A)}) \to (D(A), \|\cdot\|_{D(A)})$, which is the restriction of L_m^1 to $D(A)$. It is obvious that L_m^1 and L_m^2 have norms $\leqslant Mn!$.

Choose a $y_1 \in D(B)$ such that

$$\|J_{i,m}x - y_1\| + t \|y_1\|_{D(B)} \leqslant K(t, J_{i,m}x, \overline{D(A)}, D(B), \|\cdot\|_{D(B)}) + \varepsilon.$$

We obtain

$$K(t, x, \overline{D(A)}, D(B), \|\cdot\|_{D(B)}) \leqslant \|x - y_1\| + t \|y_1\|_{D(B)}$$
$$\leqslant \|x - J_{i,m}x\| + \|J_{i,m}x - y_1\| + t \|y_1\|_{D(B)}$$
$$\leqslant K(t, J_{i,m}x, \overline{D(A)}, D(B), \|\cdot\|_{D(B)}) + 2\varepsilon.$$

Next choose $y_2 \in D(A)$ such that

$$\|x - y_2\| + t \|y_2\|_{D(A)} \leqslant K(t, x, X, D(A), \|\cdot\|_{D(A)}) + \dfrac{\varepsilon}{Mn!}.$$

Since $J_{i,m}y_2 \in D(B)$, we have

$$
\begin{aligned}
K(t, J_{i,m}x, \overline{D(A)}, D(B), \|\cdot\|_{D(B)}) &\leqslant \|J_{i,m}x - J_{i,m}y_2\| + t\,\|J_{i,m}y_2\|_{D(B)} \\
&= \|L_m^1(x-y_2)\| + t\,\|L_m^2(y_2)\|_{D(A)} \\
&\leqslant Mn!\,(\|x-y_2\| + t\,\|y_2\|_{D(A)}) \\
&\leqslant Mn!\,K(t, x, X, D(A), \|\cdot\|_{D(A)}) + \varepsilon.
\end{aligned}
$$

Combining this and the previous inequality we obtain

$$
K(t, x, \overline{D(A)}, D(B), \|\cdot\|_{D(B)}) \leqslant Mn!\,K(t, x, X, D(A), \|\cdot\|_{D(A)}) + 3\varepsilon.
$$

Since ε is arbitrary, we complete the proof.

THEOREM 3.10. *If $A \in I_n$, then the following are equivalent for $0 < \beta \leqslant 1$ and $x \in \overline{D(A)}$:*

(a) $\left\| \dfrac{n!}{t^n}\,T(t)\,x - x \right\| = O(t^\beta)$ $(t \to 0^+)$;

(b) $\left\| \dfrac{(n+1)!}{t^{n+1}} \displaystyle\int_0^t T(u)\,x\,du - x \right\| = O(t^\beta)$ $(t \to 0^+)$;

(c) $\left\| \dfrac{n!}{t} \displaystyle\int_0^t \dfrac{T(u)\,x}{u^n}\,du - x \right\| = O(t^\beta)$ $(t \to 0^+)$;

(d) $K(t, x, X, D(A), \|\cdot\|_{D(A)}) = O(t^\beta)$ $(t \to 0^+)$.

If $\beta = 1$ the assertions (a), (b), (c), *and* (d) *are also equivalent to*

(e) $x \in \widetilde{D(B)}^{X_1}$, *where* $X_1 = \overline{D(A)}$;

(f) $x \in D(B)$, *if X is a reflexive Banach space.*

Proof. We only need to show that S^1, S^2, and S^3 in Lemma 3.7 also satisfy (A_4). Then from Theorem 2.2 and Lemma 3.9 we can derive the results.

To show S^1 satisfies (A_4), suppose $\|(n!/t^n)\,T(t)\,x - x\| \leqslant Ct^\beta$ for some $C > 0$ and all $0 < t \leqslant 1$. For such t let $m_t = [1/t] + 1$. Then $1/m_t < t \leqslant 1$ so that

$$
\begin{aligned}
\left\| \frac{1}{n+1}\,BJ_{1,m_t}x \right\| &= \frac{1}{n+1} \left\| m_t^{n+1}(n+1)!\,T\!\left(\frac{1}{m_t}\right)x - m_t^{n+1}\cdot\frac{1}{m_t^n}\cdot\frac{(n+1)!}{n!}\,x \right\| \\
&= m_t \left\| n!\,m_t^n\,T\!\left(\frac{1}{m_t}\right)x - x \right\| \leqslant Cm_t \cdot \left(\frac{1}{m_t}\right)^\beta \\
&= C\left(1 + \left[\frac{1}{t}\right]\right)^{1-\beta} \leqslant C\left(\frac{2}{t}\right)^{1-\beta} = C2^{1-\beta}\cdot t^{\beta-1}
\end{aligned}
$$

and

$$\|J_{1, m_t} x - x\| = \left\| m_t^{n+1}(n+1)! \int_0^{1/m_t} T(u) \, x \, du - x \right\|$$

$$\leq m_t^{n+1}(n+1) \int_0^{1/m_t} u^n \left\| \frac{n! \, T(u) \, x}{u^n} - x \right\| du$$

$$\leq (n+1) \, m_t^{n+1} \int_0^{1/m_t} C u^{n+\beta} \, du$$

$$\leq C(n+1) \frac{m_t^{-\beta}}{(n+1+\beta)} < C \frac{(n+1)}{n+1+\beta} t^\beta. \tag{3}$$

Hence, S^1 satisfies (A_4) with $C_1 = C2^{1-\beta}$ and $C_2 = ((n+1)/(n+1+\beta)) \, C$.

To show that S^2 satisfies (A_4), suppose $\|((n+1)!/t^{n+1}) \int_0^t T(u) \, x \, du - x\| \leq Ct^\beta$ for some $C > 0$ and all $0 < t \leq 1$. Let $m_t = [1/t] + 1$. Then $1/m_t < t$ and

$$\frac{1}{n+2} \|BJ_{2, m_t} x\| = \frac{1}{n+2} \left\| B(n+2)! \, m_t^{n+2} \int_0^{1/m_t} \int_0^u T(s) \, x \, ds \, du \right\|$$

$$= \left\| (n+1)! \, m_t^{n+2} \int_0^{1/m_t} \left(T(u) \, x - \frac{u^n}{n!} x \right) du \right\|$$

$$\leq m_t \left\| (n+1)! \, m_t^{n+1} \int_0^{1/m_t} T(u) \, x \, du - x \right\|$$

$$\leq C m_t \left(\frac{1}{m_t} \right)^\beta \leq C2^{1-\beta} t^{\beta-1},$$

and

$$\|J_{2, m_t} x - x\| = \left\| (n+2)! \, m_t^{n+2} \int_0^{1/m_t} \int_0^u T(s) \, x \, ds \, du - x \right\|$$

$$\leq (n+2) \, m_t^{n+2} \int_0^{1/m_t} u^{n+1} \left\| \frac{(n+1)!}{u^{n+1}} \int_0^u T(s) \, x \, ds - x \right\| du$$

$$\leq C \frac{(n+2)}{n+2+\beta} m_t^{-\beta} < C \frac{(n+2)}{n+2+\beta} t^\beta. \tag{4}$$

Hence S^2 satisfies (A_4) with $C_1 = C2^{1-\beta}$ and $C_2 = C(n+2)/(n+2+\beta)$.

To show that S^3 satisfies (A_4), we use integration by parts:

$$\frac{1}{t} \int_0^t \frac{T(u) \, x}{u^n} \, du = \frac{1}{t^{n+1}} \int_0^t T(u) \, x \, du + \frac{n}{t^{n+1}} \int_0^t u^{n-1} \int_0^u \frac{T(s)}{s^n} x \, ds \, du.$$

Suppose $\|(n!/t) \int_0^t (T(u) \, x/u^n) \, du - x\| \leqslant Ct^\beta$ for some $C > 0$ and all $0 < t \leqslant 1$. Let $m_t = [1/t] + 1$. Then $1/m_t < t$ and

$$
\begin{aligned}
\left\| \frac{1}{2(n+1)} BJ_{3, m_t} x \right\| &= \frac{1}{2(n+1)} (n+2)! \, m_t \left\| m_t^{n+1} \int_0^{1/m_t} B \int_0^u T(s) \, x \, ds \, du \right\| \\
&= \frac{n+2}{2(n+1)} m_t \left\| (n+1)! \, m_t \int_0^{1/m_t} \frac{T(u) \, x}{u^n} \, du - (n+1) \, x \right. \\
&\qquad \left. - n(n+1)(n!) \, m_t^{n+1} \int_0^{1/m_t} u^{n-1} \int_0^u \frac{T(s)}{s^n} x \, ds \, du + nx \right\| \\
&\leqslant \frac{1}{2} (n+2) \, m_t \left\| n! \, m_t \int_0^{1/m_t} \frac{T(u) \, x}{u^n} \, du - x \right\| \\
&\qquad + \frac{1}{2} n(n+2) \, m_t^{n+2} \int_0^{1/m_t} u^n \left\| \frac{n!}{u} \int_0^u \frac{T(s) \, x}{s^n} \, ds - x \right\| du \\
&\leqslant \frac{1}{2} C(n+2) \, m_t^{1-\beta} + \frac{1}{2} (n+2) \, m_t^{1-\beta} C \\
&\leqslant C(n+2) \, 2^{1-\beta} t^{\beta-1},
\end{aligned}
$$

and

$$
\begin{aligned}
\| J_{3, m_t} &- x \| \\
&= \left\| (n+2)! \, m_t^{n+2} \int_0^{1/m_t} \int_0^u T(s) \, x \, ds \, du - x \right\| \\
&= \left\| (n+1)(n+2) \, m_t^{n+2} \int_0^{1/m_t} n! \, u^n \int_0^u \frac{T(s) \, x}{s^n} \, ds \, du - (n+1) \, x \right. \\
&\qquad \left. - (n+1)(n+2) \, m_t^{n+2} \int_0^{1/m_t} n \cdot n! \int_0^u s^{n-1} \int_0^s \frac{T(r) \, x}{r^n} \, dr \, ds \, du + nx \right\| \\
&\leqslant (n+1)(n+2) \, m_t^{n+2} \int_0^{1/m_t} u^{n+1} \left\| \left(\frac{n!}{u} \right) \int_0^u \frac{T(s) \, x}{s^n} \, ds - x \right\| du \\
&\qquad + n(n+1)(n+2) \, m_t^{n+2} \int_0^{1/m_t} \int_0^u s^n \left\| \left(\frac{n!}{s} \int_0^s \frac{T(r) \, x}{r^n} \, dr - x \right) \right\| ds \, du \\
&\leqslant C \frac{(n+1)(n+2) \, m_t^{-\beta}}{(n+2+\beta)} + C \frac{n(n+1)(n+2) \, m_t^{-\beta}}{(n+1+\beta)(n+2+\beta)} \\
&< 2C \frac{(n+1)(n+2)}{(n+2+\beta)} t^\beta. \tag{5}
\end{aligned}
$$

Hence S^3 satisfies (A_4) with $C_1 = C(n+2) 2^{1-\beta}$, and $C_2 = 2C(n+1)(n+2)/(n+2+\beta)$.

COROLLARY 3.11. *If X is reflexive, then every operator A in I_n on X is densely defined.*

Proof. Suppose that there is a non-densely defined operator A in I_n on X. (f) of Theorem 3.10 shows $D(A) = D(B)$. In fact, for $x \in D(A)$,

$$\left\| \frac{1}{t} \left(\frac{n!\, T(t)\, x}{t^n} - x \right) \right\| \leqslant \frac{n!}{t^{n+1}} \int_0^t \| T(u)\, Ax \|\, du$$

will be bounded as $t \to 0^+$. We obtain $D(A) \subset \widetilde{D(\widetilde{B})}^{X_1} = D(B) \subset D(A)$. Since B is the part of A in $\overline{D(A)}$, this implies that the range of $\lambda - A$ is contained in $\overline{D(A)} \neq X$, which means that the resolvent set $\rho(A)$ of A is empty. This is impossible even if $T(\cdot)$ is not exponentially bounded [12]. ∎

The following corollary can be shown (see also [18, VII.4, Corollary 1′, p. 218]).

COROLLARY 3.12. *If X is reflexive, then every $A \in G(M, w)$ on X is densely defined and hence generates a C_0-semigroup.*

Proof. In [11] Kellerman and Hieber showed that if A satisfies the Hille–Yosida condition, then A generates a Lipschitz continuous one-time integrated semigroup, i.e., $A \in F_0 \subset I_1$. Then we derive the result immediately from Corollary 3.11 and the Hille–Yosida theorem. ∎

THEOREM 3.13. *Let $A \in I_n$. Then A is unbounded if and only if for each $0 < \beta < 1$ there exists $x^*_{\beta,1}$, $x^*_{\beta,2}$, and $x^*_{\beta,3}$ such that*

(i) $\left\| \dfrac{n!}{t^n}\, T(t)\, x^*_{\beta,1} - x^*_{\beta,1} \right\| \begin{cases} = O(t^\beta) \\ \neq \circ(t^\beta) \end{cases}$ $(t \to 0^+);$

(ii) $\left\| \dfrac{(n+1)!}{t^{n+1}} \displaystyle\int_0^t T(u)\, x^*_{\beta,2}\, du - x^*_{\beta,2} \right\| \begin{cases} = O(t^\beta) \\ \neq \circ(t^\beta) \end{cases}$ $(t \to 0^+);$

(iii) $\left\| \dfrac{n!}{t} \displaystyle\int_0^t \dfrac{T(u)\, x^*_{\beta,3}}{u^n}\, du - x^*_{\beta,3} \right\| \begin{cases} = O(t^\beta) \\ \neq \circ(t^\beta) \end{cases}$ $(t \to 0^+).$

Proof. By Theorem 2.4, we only need to show that S^1, S^2, and S^3 satisfy (A_5). If in (3), (4), and (5) we replace $\| (n!\, x/t^n)\, T(t)\, x - x \| \leqslant Ct^\beta,$

$\|((n+1)!/t^{n+1}) \int_0^t T(u) x \, du - x\| \leqslant Ct^\beta$ and $\|(n!/t) \int_0^t (T(u) x/u^n) \, du - x\| \leqslant Ct^\beta$ with $\|(n!/t^n) T(t) - I\| \|x\| \leqslant \varepsilon \|x\|$, $\|((n+1)!/t^{n+1}) \int_0^t T(u) \, du - I\| \|x\| \leqslant \varepsilon \|x\|$ and $\|(n!/t) \int_0^t (T(u)/u^n) \, du - I\| \|x\| \leqslant \varepsilon \|x\|$ for $\varepsilon > 0$, respectively, then computations similar to those in (3), (4), and (5) show that $\|J_{1,m} x - x\| \leqslant \varepsilon \|x\|$, $\|J_{2,m} x - x\| \leqslant \varepsilon \|x\|$, and $\|J_{3,m} x - x\| \leqslant (2n+1) \varepsilon \|x\|$ respectively. Hence S^1, S^2, and S^3 really satisfy (A_5).

If not only $A \in I_n$ but also the n-times integrated semigroup $T(\cdot)$ satisfies $\|T(t)\| \leqslant Mt^n$ for all $t \geqslant 0$, then $(0, \infty) \subset \rho(A)$ and $\|R(\lambda; A)\| = \|\int_0^\infty \lambda^n e^{-\lambda t} T(t) \, dt\| \leqslant n! \, M/\lambda$. In this case we obtain the following theorems.

THEOREM 3.14. *If A generates an n-times integrated semigroup $T(\cdot)$ with $\|T(t)\| \leqslant Mt^n$ for $t \geqslant 0$, then*

(a) $\|\lambda R(\lambda; A) x - x\| \to 0$ *as* $\lambda \to \infty$ *if and only if* $x \in \overline{D(A)}$;

(b) $\lambda(\lambda R(\lambda; A) x - x) \to Bx$ *as* $\lambda \to \infty$ *for* $x \in D(B)$.

Proof. (a) The necessity is easy to see from the fact that the range of $\lambda R(\lambda; A)$ is contained in $D(A)$. Conversely, if $x \in D(A)$ then $\|\lambda R(\lambda; A) x - x\| = \|R(\lambda; A) Ax\| \leqslant (n! \, M/\lambda) \|Ax\|$. Letting $\lambda \to \infty$ and using the uniform boundedness of $R(\lambda; A)$ we derive the sufficiency. (b) If $x \in D(B)$, then $Ax \in \overline{D(A)}$ so that $\lambda(\lambda R(\lambda; A) x - x) = \lambda R(\lambda; A) Ax$ converges to Ax by (a).

We define $J_{4,m}$ from X into X by $J_{4,m} x := mR(m; A) x$ for $m > 0$, and define $S^4 = \{\lambda R(\lambda; A), J_{4,m}, B\}$. From Theorem 3.14 it is easy to check that S^4 satisfies the hypotheses of Theorem A on the Banach space $\overline{D(A)}$.

THEOREM 3.15. *If A generates a n-times integrated semigroup $\{T(t); t \geqslant 0\}$ with $\|T(t)\| \leqslant Mt^n$ for $t \geqslant 0$, then for $0 < \beta \leqslant 1$ and $x \in X_1 = \overline{D(A)}$ the following conditions are equivalent*:

(a) $K(1/\lambda, x, \overline{D(A)}, B, \|\cdot\|_{D(B)}) = O(\lambda^{-\beta})$ $(\lambda \to \infty)$;

(b) $\|\lambda R(\lambda; A) x - x\| = O(\lambda^{-\beta})$ $(\lambda \to \infty)$;

For the particular case $\beta = 1$, (a), (b) are also equivalent to

(c) $x \in \widetilde{D(\widetilde{B})}^{X_1}$;

(d) $x \in D(B)$, *if X is a reflexive space.*

Proof. It is sufficient to show that S^4 satisfies (A_4). If $\|\lambda R(\lambda; A) x - x\| \leqslant C\lambda^{-\beta}$ for $\lambda \geqslant K \in N$, then for $m = [\lambda]$, $\|BJ_{4,m} x\| = m \|mR(m; A) x - x\| \leqslant Cm^{1-\beta} \leqslant C\lambda^{1-\beta}$ and $\|J_{4,m} x - x\| = \|mR(m; A) x - x\| \leqslant C\lambda^{-\beta}$. Hence, S^4 satsify (A_4) with $C = C_1 = C_2$.

THEOREM 3.16. *Let A be the generator of an n-times integrated semi-group $\{T(t); t \geq 0\}$ with $\|T(t)\| \leq Mt^n$ for $t \geq 0$. Then A is unbounded if and only if for each $0 < \beta < 1$ there exists $x_{\beta,4}^* \in \overline{D(A)}$ such that*

$$\|\lambda R(\lambda; A) x_{\beta,4}^* - x_{\beta,4}^*\| \begin{cases} = O(\lambda^{-\beta}) \\ \neq \circ(\lambda^{-\beta}) \end{cases} \qquad (\lambda \to \infty).$$

Proof. It is clear that S^4 satisfies (A_4).

Remarks. (1) Because of Lemma 3.9, Theorems 3.10 and 3.15 still hold when B is replaced by A.

(2) When $n = 0$, Theorems 3.8 and 3.10 reduce to Butzer and Berens' results in [3, Chapter II] and [2]; Theorem 3.13 becomes the cited result of Butzer *et al.* [9, p. 441]; Theorems 3.14, 3.15, and 3.16 reduce to some results in [7] and [5].

4. APPLICATIONS TO *N*-TIMES INTEGRATED COSINE FUNCTIONS

A strongly continuous family $\{C(t); t \geq 0\}$ of bounded linear operators on X is called a *n-times integrated cosine function* $(n \geq 1)$ if $C(0) = 0$ and

$$2C(t) C(s) x = \frac{1}{(n-1)!} \left\{ (-1)^n \int_0^{|s-t|} (|s-t| - u)^{n-1} C(u) x \, du \right.$$

$$+ \left[\int_0^{s+t} - \int_0^t - \int_0^s \right] (t+s-u)^{n-1} C(u) x \, du$$

$$\left. + \int_0^t (s-t+u)^{n-1} C(u) x \, du + \int_0^s (t-s+u)^{n-1} C(u) x \, du \right\}$$

for all $x \in X$ and $s, t > 0$. It is called a (0-times integrated) cosine function if

$$C(0) = I \qquad \text{and} \qquad 2C(t) C(s) = C(t+s) + C(t-s) \qquad \text{for} \quad t \geq s \geq 0.$$

$C(\cdot)$ is said to be *nondegenerate* if $C(t) x = 0$ for all $t > 0$ implies $x = 0$.

The *generator A* of a nondegenerate *n*-times integrated cosine function $C(\cdot)$ is defined as:

$$x \in D(A) \qquad \text{and} \qquad Ax = y \quad \text{if and only if}$$

$$C(t) x - \frac{t^n}{n!} x = \int_0^t (t-u) C(u) y \, du \qquad \text{for} \quad t \geq 0.$$

In the case that $C(\cdot)$ is exponentially bounded, i.e., $\|C(t)\| \leqslant Me^{\omega t}$, $t \geqslant 0$, one has

$$(\omega^2, \infty) \subset \rho(A) \qquad \text{and} \qquad (\lambda^2 - A)^{-1} x = \int_0^\infty \lambda^{n-1} e^{-\lambda t} C(t) \, x \, dt$$

for $x \in X$ and $\lambda > \omega$. For properties of n-times integrated cosine functions see [15].

We say $A \in I_n$ if A generates a n-times integrated cosine function $C(\cdot)$ with $\|C(t)\| = O(t^n)$ $(t \to 0^+)$.

First, we recall some properties of n-times integrated cosine functions.

PROPOSITION 4.1.   *The generator A of a n-times integrated cosine function $C(\cdot)$ is a closed operator with the following properties*:

(a)   *If $x \in D(A)$, then $C(t) x \in D(A)$ and $AC(t) x = C(t) Ax$ for $t \geqslant 0$;*

(b) $\int_0^t (t-u) C(u) x \, du \in D(A)$ *and*

$$A \int_0^t (t-u) C(u) x \, du = C(t) x - \frac{t^n}{n!} x \qquad for \quad t \geqslant 0;$$

(c) $C(\cdot)$ *is uniquely determined by A;*

(d) $C(t) x \in \overline{D(A)}$ *for each $x \in X$ and $t \geqslant 0$.*

The following theorem determines which element can be approximated by $C(\cdot)$.

THEOREM 4.2.   *Let $A \in I_n$. Then $(n!/t^n) C(t) x \to x$ as $t \to 0^+$ if and only if $x \in \overline{D(A)}$.*

Proof.   The necessity follows from (d) of Proposition 4.1. Conversely, since $\|C(t)\| \leqslant Mt^n$ as $t \to 0^+$ for some $M > 0$, it is sufficient to show that $(n!/t^n) C(t) x$ converges to x as $t \to 0^+$ for $x \in D(A)$. Let $x \in D(A)$. Then

$$\left\| \frac{n!}{t^n} C(t) x - x \right\| = \left\| \frac{n!}{t^n} \int_0^t (t-u) C(u) Ax \, du \right\|$$

$$\leqslant \frac{n!}{t^n} \int_0^t (t-u) \, \|C(u) Ax\| \, du$$

$$\leqslant \frac{n!}{(n+1)(n+2)} Mt^2 \, \|Ax\|$$

for sufficiently small t, so that letting $t \to 0^+$ we complete the proof.

COROLLARY 4.3. *If $A \in I_n$ and $x \in \overline{D(A)}$, then $((n+2)!/t^{n+2}) \int_0^t (t-u) C(u) x \, du \to x$ as $t \to 0^+$.*

Proof. It follows immediately from Theorem 4.2.

Let B be the part of A in $\overline{D(A)}$.

LEMMA 4.4. *If $A \in I_n$ then B is densely defined from $\overline{D(A)}$ into $\overline{D(A)}$. Moreover, for each $x \in D(B)$ we have*

(a) $\displaystyle \lim_{t \to 0^+} \frac{1}{t^2} \left(\frac{n!}{t^n} C(t) x - x \right) = \frac{1}{(n+1)(n+2)} Bx;$

(b) $\displaystyle \lim_{t \to 0^+} \frac{1}{t^2} \left(\frac{(n+2)!}{t^{n+2}} \int_0^t (t-u) C(u) x \, du - x \right) = \frac{1}{(n+3)(n+4)} Bx.$

Proof. Let $x \in D(A)$. Then $(n!/t^n) C(t) x \in D(A)$ and $A(n!/t^n) C(t) x = (n!/t^n) C(t) Ax \in \overline{D(A)}$, so that $(n!/t^n) C(t) x \in D(B)$. Since, by Theorem 4.2 we obtain $(n!/t^n) C(t) x \to x$ as $t \to 0^+$, it follows that $D(A) \subset \overline{D(B)} \subset \overline{D(A)}$.
To show (a), let $x \in D(B)$. Then we obtain

$$\left\| \frac{1}{t^2} \left(\frac{n!}{t^n} C(t) x - x \right) - \frac{1}{(n+1)(n+2)} Bx \right\|$$

$$= \left\| \frac{n!}{t^{n+2}} \int_0^t (t-u) C(u) Ax \, du - \frac{1}{(n+1)(n+2)} Ax \right\|$$

$$= \left\| \frac{1}{t^{n+2}} \int_0^t \int_0^u (n! \, C(r) Ax - r^n Ax) \, dr \, du \right\|$$

$$\leqslant \frac{1}{t^{n+2}} \int_0^t \int_0^u r^n \left\| \frac{n!}{r^n} C(r) Ax - Ax \right\| dr \, du.$$

From Theorem 4.2 derive (a).
To show (b), for $x \in D(B)$ we have

$$\left\| \frac{1}{t^2} \left(\frac{(n+2)!}{t^{n+2}} \int_0^t (t-u) C(u) x \, du - x \right) - \frac{1}{(n+3)(n+4)} Bx \right\|$$

$$\leqslant \frac{(n+1)(n+2)}{t^{n+4}} \int_0^t \int_0^u r^{n+2} \left\| \frac{n!}{r^{n+2}} C(r) x - \frac{x}{r^2} - \frac{1}{(n+1)(n+2)} Bx \right\| dr \, du.$$

Hence, (b) follows from (a).

LEMMA 4.5. *Let $A \in I_n$. We define $J_{5,m}$, $J_{6,m}$ from X into X by $J_{5,m} = (n+2)! \, m^{n+2} \int_0^{1/m} (1/m - u) \, C(u) \, x \, du$ and $J_{6,m} = (n+4)! \, m^{n+4} \int_0^{1/m} \int_0^u \int_0^s (s-r) \, C(r) \, x \, dr \, ds \, du$. Then for $i = 5, 6$,*

  (a) *$J_{i,m}$ us uniformly bounded;*

  (b) *$J_{i,m} x \in D(B)$ for $x \in \overline{D(A)}$;*

  (c) *$\lim_{m \to \infty} J_{i,m} x = x$ for $x \in \overline{D(A)}$.*

Proof. (a) and (b) are obvious from the assumption. It remains to show (c). For $i = 5$ the proof follows from Corollary 4.3. For $i = 6$ we have

$$\| J_{6,m} x - x \|$$

$$= (n+4)(n+3) \, m^{n+4} \int_0^{1/m} \int_0^u s^{n+2} \left\| \frac{(n+2)!}{s^{n+2}} \int_0^s (s-r) \, C(r) \, x \, dr - x \right\| ds \, du.$$

From Corollary 4.3 we derive the result.

Let $A \in I_n$. If we write $S^5 = \{(n!/t^n) \, C(t), \, J_{5,m}, \, (1/(n+1)(n+2)) \, B\}$ and $S^6 = \{((n+2)!/t^{n+2}) \int_0^t (t-u) \, C(u) \, du, \, J_{6,m}, \, (1/(n+3)(n+4)) \, B\}$. By (d) of Proposition 4.1, we know that the ranges of $(n!/t^n) \, C(t)$, $J_{5,m}$, $(n!/t^{n+2}) \int_0^t (t-u) \, C(u) \, du$, and $J_{6,m}$ are contained in $\overline{D(A)}$. Hence, we can summarize Theorem 4.2, Corollary 4.3, Lemma 4.4, and Lemma 4.5 with the following lemma.

LEMMA 4.6. *Let $A \in I_n$. Then S^5 and S^6 satisfy the hypotheses of Theorem A with parameter $\rho = 1/t$ for $0 < t < t_0$ and $\alpha = 2$ on the Banach space $\overline{D(A)}$.*

We derive the following theorem immediately from Theorem A.

THEOREM 4.7. *Let $A \in I_n$. Then the following assertions are equivalent*:

  (a) $\left\| \dfrac{n!}{t^n} \, C(t) \, x - x \right\| = \circ(t^2) \qquad (t \to 0^+);$

  (b) $\left\| \dfrac{(n+2)!}{t^{n+2}} \int_0^t (t-u) \, C(u) \, x \, du - x \right\| = \circ(t^2) \qquad (t \to 0^+);$

  (c) $x \in N(B) = N(A).$

THEOREM 4.8. *Let $A \in I_n$. The following assertions are equivalent for $0 < \beta \leqslant 2$ and $x \in \overline{D(A)}$:*

(a) $\left\| \dfrac{n!}{t^n} C(t) \, x - x \right\| = O(t^\beta) \qquad (t \to 0^+);$

(b) $\left\| \dfrac{(n+2)!}{t^{n+2}} \int_0^t (t-u) \, C(u) \, x \, du - x \right\| = O(t^\beta)(t \to 0^+);$

(c) $K(t^2, x, X, D(A), \|\cdot\|_{D(A)}) = O(t^\beta) \qquad (t \to 0^+).$

If $\beta = 2$ the assertions are also equivalent to

(d) $x \in \widetilde{D(B)}^{X_1} = \widetilde{D(A)}^{X_1},$ *where* $X_1 = \overline{D(A)};$

(e) $x \in D(B) = D(A),$ *if X is a reflexive Banach space.*

Proof.   We only need to show that S^5 and S^6 satisfy (A_4). Then the assertions follow from Theorem 2.2 with an analogue of Lemma 3.9. To show that S^5 satisfies (A_4), suppose that $\|(n!/t^n) \, C(t) \, x - x\| \leqslant Ct^\beta$ for $C > 0$ and all $0 < t \leqslant 1$. Let $m_t = [1/t] + 1$. Then $1/m_t < t$, $m_t \leqslant 2/t$, and

$$\left\| \frac{1}{(n+1)(n+2)} \, BJ_{5, \, m_t} x \right\|$$

$$= \frac{(n+2)!}{(n+1)(n+2)} \left\| Bm_t^{n+2} \int_0^{1/m_t} \left(\frac{1}{m_t} - u \right) C(u) \, x \, du \right\|$$

$$= m_t^{n+2} \left\| n! \, C\left(\frac{1}{m_t} \right) x - \frac{x}{m_t^n} \right\| \leqslant m_t^2 \left\| n! \, m_t^n \, C\left(\frac{1}{m_t} \right) x - x \right\|$$

$$\leqslant Cm_t^{2-\beta} \leqslant C2^{2-\beta} t^{\beta-2}$$

and

$$\|J_{5, \, m_t} x - x\| = \left\| (n+2)! \, m_t^{n+2} \int_0^{1/m_t} \left(\frac{1}{m_t} - u \right) C(u) \, x \, du - x \right\|$$

$$= \left\| (n+2)! \, m_t^{n+2} \int_0^{1/m_t} \int_0^u C(r) \, dr \, du - x \right\|$$

$$\leqslant (n+1)(n+2) \, m_t^{n+2} \int_0^{1/m_t} \int_0^u r^n \left\| \frac{n!}{r^n} \, C(r) \, x - x \right\| dr \, du$$

$$\leqslant C \frac{(n+1)(n+2)}{(n+1+\beta)(n+2+\beta)} \, m_t^{-\beta}$$

$$< C \frac{(n+1)(n+2)}{(n+1+\beta)(n+2+\beta)} \, t^\beta. \tag{6}$$

Hence, S^5 satisfies (A_4) with $C_1 = C2^{2-\beta}$ and $C_2 = C(n+1)(n+2)/(n+1+\beta)(n+2+\beta)$.

To show S^6 satisfies (A_4), suppose that $\|((n+2)!/t^{n+2})\int_0^t (t-u)\, C(u)\, x\, du - x\| \leqslant Ct^\beta$ for $C > 0$ and all $0 < t \leqslant 1$. Let $m_t = [1/t] + 1$. Then $1/m_t < t$, $m_t \leqslant 2/t$, and

$$\left\| \frac{1}{(n+3)(n+4)}\, BJ_{6, m_t} \right\|$$

$$= \frac{(n+4)!}{(n+3)(n+4)} \left\| m_t^{n+4} B \int_0^{1/m_t} \int_0^u \int_0^s (s-r)\, C(r)\, x\, dr\, ds\, du \right\|$$

$$\leqslant m_t^2 \left\| m_t^{n+2}(n+2)! \int_0^{1/m_t} \left(\frac{1}{m_t} - u \right) C(u)\, x\, du - x \right\|$$

$$\leqslant Cm_t^{2-\beta} \leqslant C2^{2-\beta}t^{\beta-2},$$

and

$$\|J_{6, m_t} x - x\|$$

$$= \left\| m_t^{n+4}(n+4)! \int_0^{1/m_t} \int_0^u \int_0^s (s-r)\, C(r)\, x\, dr\, ds\, du - x \right\|$$

$$\leqslant (n+3)(n+4)\, m_t^{n+4} \int_0^{1/m_t} \int_0^u s^{n+2} \left\| \frac{(n+2)!}{s^{n+2}} \int_0^s (s-r)\, C(r)\, x\, dr - x \right\| ds\, du$$

$$\leqslant C \frac{(n+3)(n+4)}{(n+3+\beta)(n+4+\beta)}\, t^\beta.$$

Hence, S^6 satisfy (A_4) with $C_1 = C2^{2-\beta}$ and $C_2 = C(n+3)(n+4)/(n+3+\beta)(n+4+\beta)$.

THEOREM 4.9. *Let $A \in I_n$. Then A is unbounded if and only if for each $0 < \beta < 2$ there exists $x_{\beta,5}^*$ and $x_{\beta,6}^*$ such that*

(i) $\left\| \dfrac{n!}{t^n}\, C(t)\, x_{\beta,5}^* - x_{\beta,5}^* \right\| \begin{cases} = O(t^\beta) \\ \neq \circ(t^\beta) \end{cases} \quad (t \to 0^+);$

(ii) $\left\| \dfrac{(n+2)!}{t^{n+2}} \int_0^t (t-u)\, C(u)\, x_{\beta,6}^*\, du - x_{\beta,6}^* \right\| \begin{cases} = O(t^\beta) \\ \neq \circ(t^\beta) \end{cases} \quad (t \to 0^+).$

Proof. By Theorem 2.4, we only need to show S^5 and S^6 satisfy (A_5). If in (6) and (7) we replace $\|(n!\,x/t^n)\,C(t)\,x-x\| \leqslant Ct^\beta$ and $\|((n+2)!/t^{n+2})\int_0^t (t-u)\,C(u)\,x\,du-x\| \leqslant Ct^\beta$ with $\|(n!/t^n)\,C(t)-I\| \|x\| \leqslant \varepsilon\,\|x\|$ and $\|((n+2)!/t^{n+2})\int_0^t (t-u)\,C(u)\,du-I\| \|x\| \leqslant \varepsilon\,\|x\|$ for $\varepsilon > 0$, respectively, then computations similar to those in (6) and (7) show that $\|J_{5,m}x-x\| \leqslant \varepsilon\,\|x\|$ and $\|J_{6,m}x-x\| \leqslant \varepsilon\,\|x\|$, respectively. Hence S^5 and S^6 really satisfy (A_5).

Moreover, if not only $A \in I_n$ but also the n-times integrated cosine function $C(\cdot)$ satisfies $\|C(t)\| \leqslant Mt^n$ for all $t \geqslant 0$, then $(0,\infty) \subset \rho(A)$ and $\|R(\lambda^2; A)\| = \|\int_0^\infty \lambda^{n-1} e^{-\lambda t} C(t)\,dt\| \leqslant n!\,M/\lambda^2$.

If we replace λ by λ^2 in Theorem 3.14, Theorem 3.15, and Theorem 3.16, then the similar proofs yield the following theorems.

THEOREM 4.10. *If A generates a n-times integrated cosine function $C(\cdot)$ with $\|C(t)\| \leqslant Mt^n$ for $t \geqslant 0$, then*

(a) $\|\lambda^2 R(\lambda^2; A)\,x-x\| \to 0$ *as* $\lambda \to \infty$ *if and only if* $x \in \overline{D(A)}$;

(b) $\lambda^2(\lambda^2 R(\lambda^2; A)\,x-x) \to Bx$ *as* $\lambda \to \infty$ *for* $x \in D(B)$.

THEOREM 4.11. *If A generates a n-times integrated cosine function $C(\cdot)$ with $\|C(t)\| \leqslant Mt^n$ for $t \geqslant 0$, then for $0 < \beta \leqslant 2$ and $x \in X_1 = \overline{D(A)}$ the following conditions are equivalent*:

(a) $K(1/\lambda^2, x, \overline{D(A)}, B, \|\cdot\|_{D(B)} = O(\lambda^{-\beta})$ $(\lambda \to \infty)$;

(b) $\|\lambda^2 R(\lambda^2; A)\,x-x\| = O(\lambda^{-\beta})$ $(\lambda \to \infty)$;

For the particular case $\beta = 2$, (a), (b) are also equivalent to

(c) $x \in \widetilde{D(B)}^{X_1} = \widetilde{D(A)}^{X_1}$;

(d) $x \in D(B) = D(A)$, *if X is a reflexive space.*

THEOREM 4.12. *Let A be the generator of a n-times integrated cosine function $C(\cdot)$ with $\|C(t)\| \leqslant Mt^n$ for $t \geqslant 0$. Then A is unbounded if and only if for each $0 < \beta < 2$ there exists a $x^*_{\beta,7} \in \overline{D(A)}$ such that*

$$\|\lambda^2 R(\lambda^2; A)\,x^*_{\beta,7} - x^*_{\beta,7}\| \begin{cases} = O(\lambda^\beta) \\ \neq \circ(\lambda^{-\beta}) \end{cases} \qquad (\lambda \to \infty).$$

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